Gauge Invariant Formulation and Bosonisation of the Schwinger Model

J. Kijowski

Center for Theor. Phys., Polish Academy of Sciences
al. Lotników 32/46, 02 - 668 Warsaw, Poland

G. Rudolph, M. Rudolph

Institut für Theoretische Physik, Univ. Leipzig
Augustusplatz 10/11, 04109 Leipzig, Germany

Abstract

The functional integral of the massless Schwinger model in $(1+1)$ dimensions is reduced to an integral in terms of local gauge invariant quantities. It turns out that this approach leads to a natural bosonisation scheme, yielding, in particular the famous "bosonisation rule" and giving some deeper insight into the nature of the bosonisation phenomenon. As an application, the chiral anomaly is calculated within this formulation.

keywords: schwinger model; gauge invariants; bosonisation; chiral anomaly
PACS: 11.15.-q; 11.10.Kk; 11.25.Sq; 11.30.Rd

1 Introduction

In recent years there has been some effort to understand the classical bosonisation results for models in (1+1) dimensions, see [1], [2] and further references therein, in terms of functional integral techniques, see [4, 5, 6], [7] and further references therein. In this Letter we apply our programme of formulating gauge theories in terms of local gauge invariant quantities, see [8], [9] and [10], to the massless Schwinger model in 2 dimensions, see [3]. We show that it leads to a natural explanation of the bosonisation phenomenon, including mass generation. We also demonstrate that our formulation provides us with the possibility to calculate the chiral anomaly in a straightforward, simple way. We stress that this is in contrast to calculations within the "ordinary" formulation, where one either uses perturbative techniques or topologically motivated regularizations (heat kernel, zeta–function), see e.g. [16].

Our approach is based upon the following ideas: First one has to analyze the algebra of Grassmann-algebra-valued invariants, which one can build from the gauge field and the anticommuting matter fields. For a systematic study of this aspect for the (most complicated) case of QCD we refer to [11]. Typically, there occurs a number of relations between invariants, which on the level of the algebra cannot be "solved", because dividing by Grassmann-algebra-valued quantities is in general not well defined. Next one has to express the Lagrangian, or at least the Lagrangian multiplied by some nonvanishing element of the above algebra, in terms of invariants. The main point of our strategy consists in reducing the original functional integral measure to a measure in terms of invariants, for details we refer to [9] and [10]. In particular, under the functional integral we are able to solve the above-mentioned relations between invariants, leading to theories

with the correct number of effective fields. Along these lines we have found, both for QED and QCD, a functional integral formulation in terms of effective bosonic fields.

The paper is organized as follows: In Section 2 we introduce some basic notations and give the definition of gauge invariants, built from the gauge potential and the (anti-commuting) spinor fields. Some algebraic relations among these invariants, necessary for our purposes, are written down. The main result of Section 3 is the functional integral completely rewritten in terms of local (bosonic) gauge invariants. Finally, in Section 4 we derive the standard bosonisation rule and calculate the chiral anomaly.

2 Basic Notations and Gauge Invariants

A field configuration of the Schwinger model [3] is given by a $U(1)$-gauge potential A_μ and a spinor field ψ^K. In what follows $\mu, \nu, \ldots = 0, 1$ denote space–time indices and $K, L, \ldots = 1, 2$ spinor indices. The spinor field is represented by

$$\psi^K = \begin{pmatrix} \phi \\ \varphi \end{pmatrix}, \qquad \psi^\dagger_K = (\phi^*, \varphi^*), \tag{2.1}$$

where the star denotes complex conjugation. The 4 elements (ψ^K, ψ^*_K) anticommute and (pointwise) generate a 16-dimensional Grassmann-algebra.

The Dirac–matrices in Minkowski space with metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1)$ are taken as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

The completely antisymmetric symbol in 2 dimensions is denoted by $\epsilon_{\mu\nu}$, with $\epsilon_{01} = 1$.

The starting point for quantum theory is the following (formal) functional integral:

$$Z = \int \prod_x \mathrm{d}A \, \mathrm{d}\psi \, \mathrm{d}\bar\psi \, e^{\mathrm{i} \int \mathrm{d}^2 x \, \mathcal{L}}, \tag{2.3}$$

with the Lagrangian

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{mat} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \mathrm{Im} \left\{ \overline{\psi_K} \, (\gamma^\mu)^K{}_L \, (D_\mu \psi^L) \right\}. \tag{2.4}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the field strength, $D_\mu \psi^K = \partial_\mu \psi^K + ie \, A_\mu \psi^K$ the covariant derivative and $\overline{\psi_L} := \psi^\dagger_K (\gamma^0)^K{}_L = (\varphi^*, \phi^*)$.

Now we start to apply our approach to this model. For this purpose we define the following set of gauge invariant Grassmann-algebra valued quantities:

$$L := \varphi^* \varphi, \qquad R := \phi^* \phi, \qquad H := \phi^* \varphi, \tag{2.5}$$

$$B_\mu := \tfrac{1}{2} \mathrm{Im} \left\{ H^* \left(\phi^* (D_\mu \varphi) + \varphi (D_\mu \phi^*) \right) \right\}. \tag{2.6}$$

Obviously, L, R and H are (pointwise) elements of rank two, obeying the following identity:

$$L R = -|H|^2. \tag{2.7}$$

2

Moreover, $|H|^2 = HH^*$ is a nonvanishing element of (pointwise) maximal rank. The vector current is defined by

$$J^\mu := \overline{\psi_K} \, (\gamma^\mu)^K{}_L \, \psi^L = \begin{pmatrix} R + L \\ R - L \end{pmatrix}. \tag{2.8}$$

In 2 dimensions we have the relation $J_5^\mu = \epsilon^{\mu\nu} J_\nu$. Thus, we get for the axial current

$$J_5^\mu = \begin{pmatrix} R - L \\ R + L \end{pmatrix}. \tag{2.9}$$

Finally, for B_μ we have

$$B_\mu = e \, |H|^2 \, A_\mu + \tfrac{1}{4i} \, \{\varphi^* \phi^* \, \partial_\mu(\varphi\phi) + \varphi\phi \, \partial_\mu(\phi^*\varphi^*)\}. \tag{2.10}$$

3 Functional Integral

In order to be able to calculate expectation values of physical quantities one usually applies the Faddeev–Popov gauge fixing procedure, see [12] and [13], to the functional integral (2.3). Here we use a completely different procedure, proposed in [9] and [10], which consists in reducing (2.3) to an integral in terms of the above defined gauge invariants.

For that purpose we introduce the following auxiliary gauge invariants:

$$C_\mu := \phi^* \, (D_\mu\varphi), \qquad E_\mu := \varphi^* \, (D_\mu\varphi), \qquad F_\mu := \phi^* \, (D_\mu\phi), \tag{3.1}$$

which are subject to the relations:

$$\begin{align}
H^* C_\mu &= \mathrm{i} \, B_\mu + \tfrac{1}{2} \, H^*(\partial_\mu H) - \tfrac{1}{4} \, R(\partial_\mu L) + \tfrac{1}{4} \, L(\partial_\mu R), \tag{3.2} \\
R E_\mu &= -\mathrm{i} \, B_\mu - \tfrac{1}{2} \, H^*(\partial_\mu H) + \tfrac{1}{4} \, R(\partial_\mu L) - \tfrac{1}{4} \, L(\partial_\mu R), \tag{3.3} \\
L F_\mu &= -\mathrm{i} \, B_\mu + \tfrac{1}{2} \, H^*(\partial_\mu H) - \tfrac{1}{4} \, R(\partial_\mu L) + \tfrac{1}{4} \, L(\partial_\mu R) + \tfrac{1}{2}\partial_\mu(LR). \tag{3.4}
\end{align}$$

One easily proves the following identities:

$$\begin{align}
e \, |H|^2 F_{\mu\nu} &= -\mathrm{i} \, \partial_{[\mu}(|H|^2 C_{\nu]}) + \mathrm{i} \, C_{[\mu}^* C_{\nu]} - \mathrm{i} \, (\partial_{[\mu} R) \, E_{\nu]}, \tag{3.5} \\
\mathcal{L}_{mat} &= -\tfrac{1}{2} \, \mathrm{tr} \, (\gamma^0 \gamma^\mu) \, \{(\eta_{\mu\nu} + \epsilon_{\mu\nu}) \, \mathrm{Im} F^\nu + (\eta_{\mu\nu} - \epsilon_{\mu\nu}) \, \mathrm{Im} E^\nu\} \\
&\equiv -\mathrm{Im} \, \{E_0 + F_0 - E_1 + F_1\}. \tag{3.6}
\end{align}$$

Observe, that (3.5) is an identity on the level of maximal rank in the Grassmann–algebra. Dividing by a non–vanishing element of maximal rank is a well–defined operation giving a c–number. Thus, we can express the Lagrangian (2.4) completely in terms of gauge invariants.

With the identities (3.2)–(3.6) we are now able to reduce the functional integral (2.3) to an integral over gauge invariants. For that purpose we make use of the following notion of the δ–distribution on superspace (see [17])

$$\delta(u - U) = \int \mathrm{d}\xi \, e^{2\pi\mathrm{i}\xi(u-U)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta^{(n)}(u) U^n. \tag{3.7}$$

3

Here u is a c–number variable and U an element of the Grassmann–algebra built from the matter fields ψ and ψ^*. From this definition we get immediately

$$1 \equiv \int \mathrm{d}u \, \delta(u - U). \tag{3.8}$$

Thus, by inserting identites of the form (3.8) under the functional integral we introduce for each Grassmann–algebra–valued gauge invariant $(L, R, H, B_\mu, J_\mu, J_\mu^5)$ c–number variables $(l, r, h, b_\mu, j_\mu, j_\mu^5)$, which we will call c–number mates. These mates are by definition gauge invariant: r and l are real scalar fields, h is a complex scalar field and b_μ, j_μ and j_μ^5 are real covector fields. Next we apply the procedure developed in [9] and [10]: First we use (3.5) and (3.6) under the functional integral to express the Lagrangian in terms of gauge invariants. Next we eliminate the auxiliary invariants by the help of (2.8)–(2.9) and (3.2)–(3.4). We are left with a functional integral, where the original gauge dependent field configuration (ψ, A_μ) occurs only in the integral measure. It can be integrated out leaving us with the (reduced) functional integral in terms of gauge invariants (for details we refer to [9] and [10]). Finally we stress that, due to (2.8)–(2.9), r, l and j_μ can be expressed by j_μ^5. Thus, for this model we get a reduction of the functional integral (2.3) to an integral in terms of the gauge invariant set $(v_\mu, j_5^\mu, |h|, \theta)$, where $h := |h|e^{\mathrm{i}\theta}$ and $v_\mu := \frac{b_\mu}{e|h|^2}$:

$$Z = \int \prod_x \left\{ \mathrm{d}v_\mu \mathrm{d}j_5^\mu \, \mathrm{d}|h|^2 \, \mathrm{d}\theta \, K[j_5^\mu, |h|] \right\} \, e^{\mathrm{i} \int \mathrm{d}^2 x \, \mathcal{L}[v_\mu, j_5^\mu, \theta]}, \tag{3.9}$$

with

$$K[j_5^\mu, |h|] = \frac{1}{2\pi} \left(\frac{\delta^2}{\delta j_\mu^5 \, \delta j_5^\mu} - \frac{1}{4} \frac{\delta^2}{\delta|h|^2} + \frac{1}{4|h|} \frac{\delta}{\delta|h|} \right) \delta(j_\mu^5)\delta(|h|^2), \tag{3.10}$$

$$\mathcal{L}[v_\mu, j_\mu^5, \theta] = -\tfrac{1}{4} \left(\partial_{[\mu} v_{\nu]} \right)^2 - e \, \epsilon^{\mu\nu} \, v_\mu \, j_\nu^5 + \tfrac{1}{2} j_\mu^5 \left(\partial^\mu \theta \right). \tag{3.11}$$

We stress that $\mathrm{d}\theta$ is the Haar–measure on S^1.

The effective Lagrangian (3.11) leads to the following set of classical field equations in terms of gauge invariants:

$$(\partial_\mu j_5^\mu) = 0, \tag{3.12}$$

$$v^\mu = \tfrac{1}{2e} \, \epsilon^{\mu\nu} \, (\partial_\nu \theta), \tag{3.13}$$

$$(\partial_\mu \partial^{[\mu} v^{\nu]}) = e \, \epsilon^{\nu\mu} \, j_\mu^5. \tag{3.14}$$

Formula (3.12) means that, on the classical level, the axial current is conserved. Moreover, from equation (3.14) we get $(\partial_\mu j^\mu) = 0$, i.e. also the vector current is conserved. Finally, equation (3.13) leads to

$$(\partial_\mu v^\mu) = 0. \tag{3.15}$$

We see that, as in the Faddeev–Popov procedure, a nontrivial singular integral kernel $K[j_5^\mu, |h|]$ occurs in the functional integral (3.9). Obviously, the quantity $|h|$ can

be integrated out, because it does not occur in (3.11). The remaining kernel $K[j_5^\mu] = \frac{1}{2\pi}\left(\frac{\delta^2}{\delta j_\mu^5 \delta j_5^\mu}\right)\delta(j_5^\mu)$ can be treated similarly as in the Faddeev–Popov procedure: it can be averaged with a Gaussian measure, or more generally, with a sum of moments of a Gaussian measure. After this "regularization" a free parameter, say α_j, characterizing the Gaussian measure, enters the theory. In the Faddeev–Popov approach different values of this parameter correspond to different gauge fixings. Here, as will be seen in what follows, this parameter is fixed, e.g. by the physical requirement to give the correct chiral anomaly. After this "averaging" procedure we get

$$Z = \mathcal{N} \int \prod_x \{\mathrm{d}v_\mu \mathrm{d}j_5^\mu \, \mathrm{d}\theta\} \; e^{\mathrm{i} \int \mathrm{d}^2 x \, \mathcal{L}[v_\mu, j_5^\mu, \theta]}, \tag{3.16}$$

where

$$\mathcal{L}[v_\mu, j_\mu^5, \theta] = -\tfrac{1}{4}\left(\partial_{[\mu} v_{\nu]}\right)^2 - e\,\epsilon^{\mu\nu}\,v_\mu\,j_\nu^5 + \tfrac{1}{2}\,j_\mu^5\,(\partial^\mu\theta) - \tfrac{1}{2\alpha_j}\,j_5^\mu j_\mu^5. \tag{3.17}$$

Here, $\mathcal{N}$ is an infinite normalization constant, which we omit in what follows. We stress that the same procedure works also for the standard generating functional.

4   Standard Bosonisation Rule and Chiral Anomaly

Since j_μ^5 does not occur as a dynamical field a further reduction of (3.16) is possible, namely we can carry out the Gaussian integration over j_μ^5. This leads to:

$$Z = \int \prod_x \{\mathrm{d}v_\mu \, \mathrm{d}\theta\} \; e^{\mathrm{i} \int \mathrm{d}^2 x \, \mathcal{L}[v_\mu, \theta]}, \tag{4.1}$$

where

$$\mathcal{L}[v_\mu, \theta] = -\tfrac{1}{4}\left(\partial_{[\mu} v_{\nu]}\right)^2 - \tfrac{\alpha_j\,e^2}{2}\,v_\mu v^\mu - \tfrac{\alpha_j\,e}{2}\,\epsilon^{\mu\nu}\,v_\mu(\partial_\nu\theta) + \tfrac{\alpha_j}{8}\,(\partial_\mu\theta)(\partial^\mu\theta). \tag{4.2}$$

After performing the reparameterisation $\theta(x) \to 2\sqrt{\pi}\theta(x)$ as well as fixing the parameter $\alpha_j = \tfrac{1}{\pi}$ we obtain:

$$\mathcal{L} = -\tfrac{1}{4}\left(\partial_{[\mu} v_{\nu]}\right)^2 - \tfrac{e^2}{2\pi}\,v_\mu v^\mu - \tfrac{e}{\sqrt{\pi}}\,\epsilon^{\mu\nu}\,v_\mu(\partial_\nu\theta) + \tfrac{1}{2}\,(\partial_\mu\theta)(\partial^\mu\theta). \tag{4.3}$$

This result has to be compared with other functional integral approaches explaining the bosonisation phenomenon, see [4, 5, 6]. Essentially, our procedure leads to the same result, but with the following advantages: Within our approach, instead of the electromagnetic potential in a certain gauge, the gauge invariant field v_μ occurs. It becomes – automatically – massive, with mass $m_v = \frac{e}{\sqrt{\pi}}$. Thus, we see a dynamical Higgs mechanism characteristic for the Schwinger model, see [14], [15], [16] and references therein. Moreover, the field θ, which in other approaches has to be rather introduced by hand, shows up as the "phase" of a gauge invariant combination of the original fermionic fields. This

gives – in our opinion – a deeper insight into the bosonisation phenomenon. Comparing the couplings of v_μ in (3.17) and (4.3), we read off the famous "bosonisation rule"

$$\frac{1}{\sqrt{\pi}}(\partial^\mu\theta) = j_5^\mu. \tag{4.4}$$

Next, we show that within our approach the chiral anomaly can be easily calculated, justifying – in particular – the above made choice $\alpha_j = \frac{1}{\pi}$. For this purpose, we treat v_μ as an external field. Due to the bosonisation rule (4.4), the chiral anomaly should be given by the vacuum expectation value

$$< \frac{1}{\sqrt{\pi}}(\partial^\mu\theta) > = \frac{1}{Z[v_\mu]} \int d\theta \, \frac{1}{\sqrt{\pi}}(\partial^\mu\theta) \, e^{i\int d^2x\tilde{\mathcal{L}}}, \tag{4.5}$$

where $Z[v_\mu] = \int d\theta \, e^{i\int d^2x\tilde{\mathcal{L}}}$ and $\tilde{\mathcal{L}} = -\frac{e^2}{2\pi}v_\mu v^\mu - \frac{e}{\sqrt{\pi}}\epsilon^{\mu\nu}v_\mu(\partial_\nu\theta) + \frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta)$. Since v_μ is an external field, i.e. it obeys the classical field equation (3.15), we can write it in the form $v_\mu = \epsilon_{\mu\nu}(\partial^\nu\Lambda)$. Thus, performing the shift $(\partial^\mu\theta) \rightarrow (\partial^\mu\theta) + \frac{e}{\sqrt{\pi}}\epsilon^{\nu\mu}v_\nu \equiv (\partial^\mu\theta) - \frac{e}{\sqrt{\pi}}(\partial^\mu\Lambda)$, which leads to a trivial change in the integration variable $\theta(x) \rightarrow \theta(x) - \frac{e}{\sqrt{\pi}}\Lambda(x)$, we get:

$$\begin{aligned}
< \frac{1}{\sqrt{\pi}}(\partial^\mu\theta) > &= \frac{1}{Z[v_\mu]} \int d\theta \, \frac{1}{\sqrt{\pi}}\left((\partial^\mu\theta) + \frac{e}{\sqrt{\pi}}\epsilon^{\nu\mu}v_\nu\right) e^{i\int d^2x\{\frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta)\}} \\
&= \frac{1}{Z[v_\mu]} \int d\theta \, \frac{1}{\sqrt{\pi}}(\partial^\mu\theta) e^{i\int d^2x\{\frac{1}{2}(\partial_\mu\theta)(\partial^\mu\theta)\}} + \frac{e}{\pi}\epsilon^{\nu\mu}v_\nu.
\end{aligned}$$

Now observe that the first integral is odd in θ and, consequently, it vanishes identically, leading to

$$< \frac{1}{\sqrt{\pi}}(\partial^\mu\theta) > = \frac{e}{\pi}\epsilon^{\nu\mu}v_\nu. \tag{4.6}$$

Taking the partial derivative gives exactly the chiral anomaly:

$$\partial_\mu < \frac{1}{\sqrt{\pi}}(\partial^\mu\theta) > \equiv < \frac{1}{\sqrt{\pi}}(\partial_\mu\partial^\mu\theta) > = \frac{e}{\pi}\epsilon^{\nu\mu}(\partial_\mu v_\nu). \tag{4.7}$$

We see that within our formulation the chiral anomaly can be calculated in a straightforward way without using neither perturbative techniques nor regularization techniques of the heat kernel type. We also stress that this calculation can be – a posteriori – considered as a deeper justification of the bosonisation rule (4.4).

Moreover, we have

$$< \frac{1}{\sqrt{\pi}}\epsilon^{\nu\mu}(\partial_\nu\theta) > = \frac{e}{\pi}v^\mu. \tag{4.8}$$

Using again the classical field equation (3.15) yields

$$\partial_\mu < \frac{1}{\sqrt{\pi}}\epsilon^{\nu\mu}(\partial_\nu\theta) > = \frac{e}{\pi}(\partial_\mu v^\mu) \equiv 0. \tag{4.9}$$

Due to (4.4) and the relation between vector and axial current, equation (4.9) shows the conservation of the vector current j_μ.

Finally, let us make some remarks. If we consider the massive Schwinger model, where an additional term $-m\overline{\psi}\psi$ occurs in the Lagrangian (2.4), we get an additional term $-2m\mathrm{Re}(h) = -2m|h|\cos\theta$ in (3.3). This leads to a field theory of sine–Gordon type. Comparing with the standard result, see for instance [1] and [2], we obtain a slight modification: instead of getting a constant in front of the $\cos(\theta)$–term, we obtain $|h|$. However, since $|h|$ is a non-propagating field, it can be "averaged" to a constant – along the same lines as discussed at the end of Section 3.

Our approach also works for the 2–dimensional gauged Thirring–model (see for instance [7], where the Thirring–model and its various generalizations on curved space–time are analyzed). Applying our procedure simply leads to a modification of the coefficient of the term quadratic in j_μ^5 occuring in equation (3.17).

Similar investigations for full QED are in progress.

Acknoledgement

References

[1] S.Coleman, Phys. Rev **D 11** (1975), 2088

[2] Frenkel, J. Funct. Anal. **44** (1981), 259

[3] J.Schwinger, Phys. Rev **128** (1962), 2425

[4] P.H.Damgaard and H.B.Nielsen, Nucl. Phys. **B 385** (1992), 227

[5] P.H.Damgaard, H.B.Nielsen and R.Sollacher, CERN-TH-6959/93

[6] P.H.Damgaard, H.B.Nielsen and R.Sollacher, Phys. Lett. **B 296** (1992), 132

[7] I.Sachs and A.Wipf, Ann. Phys. (NY) **249** (1996), 380

[8] J.Kijowski and G.Rudolph, Lett. Math. Phys. **29** (1993), 103

[9] J.Kijowski, G.Rudolph and M.Rudolph, Lett. Math. Phys. **33** (1995), 139

[10] J.Kijowski, G.Rudolph and M.Rudolph, *Effective Bosonic Degrees of Freedom for One–Flavour Chromodynamics*, Ann. Inst. H. Poincaré, (in print), preprint-Nr. 22/1996, Univ. Leipzig

[11] J.Kijowski, G.Rudolph and M.Rudolph, *On the Algebra of Gauge Invariants for One–Flavour Chromodynamics*, Rep. Math. Phys., (in print)

[12] L.D.Faddeev and V.N.Popov, Phys. Lett. **B 25** (1967), 30

[13] L.D.Faddeev, A.A.Slavnov, *Gauge Fields*, The Benjamin/Cummings Publ. Comp., London 1980

[14] E.Abdalla,M.C.B.Abdalla and K.D.Rothe, *Non-Perturbative Methods in 2 dimensional Quantum Field Theory*, World Scientific, Singapore, London 1991

[15] C.Adam,R.A.Bertlmann and P.Hofer, UWThPh-1993-5

[16] R.A.Bertlmann, *Anomalies in Quantum Field Theory*, Clarendon Press, Oxford 1996

[17] F.A.Berezin, *The Method of Second Quantization*, Academic Press, New York and London 1966